Simple Theory of Norm and Action[1]

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1. Introduction

Deontic properties of actions, such as: “being obligatory”, “being permitted" or “being forbidden”, are components of norms, which are the interest’s subject of Law, Ethics, Information Technology, Telecommunication and many other fields. In the fifties of the last century, the newly-formed branch of modal logic, called deontic logic, provided formal tools to describe deontic properties, at the same time indicating relationships between them.
It should be stressed that in the first systems of deontic logic – Old System (OS) [von Wright 1951], systems K_1 and K_2 [Kalinowski, 1953] – the deontic properties have been treated as properties of actions, not “modalities”. This approach fits with the way in which the norms are being formulated, not only in natural language, but also in Law and Ethics, allowing to form such logical structures as “the action *a* is prohibited”, or “if the agent *x* is obliged to do *a*, then it is permitted to *x* to do *a*”[2].

It is well known that modern systems of deontic logic are in the large number not the logics of deontic properties but deontic modalities. Strictly speaking, they are the deontic logics of the states of affairs, sometimes understood as the outcome of actions. For instance, they are used to express that “a state of affairs *p* is permitted” or that “the agent ought to see to it that *p*” [Horty 2001]. This approach, although proved to be useful in some applications, for example in computer science – particularly in systems responsible for the security of information systems, whose security levels are described by the expression saying that a certain state of the system is permitted or forbidden – appears to be inadequate for modeling, legal and ethical issues, where a direct reference to actions and their deontic properties is required.

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[1] This paper in the large part is English translation of my paper: *Setna - prosta teoria norm i działań*, which appeared in the Polish journal *Filozofia Nauki*, nr 3-4 (63-64) in 2008. In the version presented here, I slightly modified the language of the theory and made important references to its extensions which have been made since 2008.

[2] Formal systems in which deontic properties applied to actions got their second life starting from 1982:

> “*Deontic Logic of Action* [Segerberg, 1982], an article published by K. Segerberg in 1982 was a milestone in the development of deontic logic since the 1950s when vonWright [von Wright, 1951] and Kalinowski [Kalinowski, 1953] had published their innovative deontic systems. In his article Segerberg proposed two systems: *basic open deontic logic of urn model action* (B.O.D.) and *basic closed deontic logic of urn model action* (B.C.D.). He also provided two models for both systems and proved the adequacy theorems. Segerberg’s work has become a source of inspiration for the deontic first order theories of Lokhorst [Lokhorst, 1996] and Trypuz [Trypuz, 2008], and deontic logics of action built in connection with Propositional Dynamic Logic (PDL). In the last class of systems, which are perhaps the most developed and explored deontic logics of action nowadays, we can distinguish (i) those in which deontic operators are defined in the Andersonian-Kangerian style by PDL operators and constant *V* (violation) [Meyer 1988, Dignum et al. 1996] and (ii) those which are built on PDL and in which (at least some) deontic operators are taken as primitive. The second approach may be further divided into the systems built on top of standard PDL [McCarty, 1983, Meyden, 1996] and the systems based on Boolean algebra of actions [Castro and Maibaum, 2009].” [Trypuz, Kulicki 2009]

What's more, I claim that the deontic properties of actions directly depend on other, non-deontic properties of actions[3]. For example, if some agent is unable to carry out two actions simultaneously in certain situation, it is pointless to regulate those two actions as "obligatory" in this situation, because the agent won't be "physically" able to satisfy such norms. So there is a need to return to the deontic logic of action, and expand it by relations between deontic and non-deontic properties of actions.

This article attempts to meet this need. I would like to propose a simple theory of norm and action (in short, SETNA). It is a theory inspired by the first deontic logics. SETNA's theses are all the basic laws of deontic logic in which internal operators do not appear in the scope of deontic operators. Moreover, a part of SETNA is also a theory of action, which describes the selected non-deontic properties of actions (e.g., physical or intellectual ability to perform an action by the agent or the (im)possibility of simultaneous execution of two or more actions in certain circumstances by the same agent) and formulates dependencies taking place between them and the deontic properties.

The structure of the paper is the following. In section 2, the language, axioms, definitions, and some theses of SETNA are presented. Model of SETNA is the subject of section 3. The central place in the model takes the concept of "situation" being in essence a collection of actions satisfying certain conditions. These conditions are discussed in section 3.1. In section 3.2, the soundness and completeness of SETNA with respect to the models presented in section 3.1 is proved. In section 4, an application of SETNA is shown. Finally in the conclusion some extensions of SETNA are described.

2. SETNA – as a theory of FOL

SETNA is a formal theory modeling: (i) deontic properties of actions, (ii) abilities of the agent to carry out actions and the relationships between (i) and (ii). From the logical point of view, SETNA is a theory of first-order predicate calculus (in short: FOL) with identity[4]. The language of SETNA consists of

- operators of propositional calculus (PC): $\neg, \wedge, \vee, \rightarrow, \equiv$ (negation, conjunction, disjunction, implication and equivalence, respectively)
- quantifiers: $\forall$ and $\exists$
- identity sign: $=$
- variables representing actions: $a, b, c, a_1, \ldots$
- constants naming actions: $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a_1}, \ldots$
- terms: $t_1, t_2, t_3, \ldots$
- two specific binary relations: $\approx$ (coexistence/co-occurrence of two actions) and S (parallel execution of two actions)
- two unary deontic predicates: O (obligation to do) and F (forbiddance to do)

SETNA has eight specific axioms that are introduced and discussed below. It is assumed that theses of SETNA relate to one and the same agent. Hence, there is no direct reference to the agents in the theory. Here are the axioms:

[3] Formal non-deontic properties of actions have been studied by the author in [Troquard et al. 2006, Trypuz 2007, Trypuz, Vieu 2007, Trypuz et al. 2007].

[4] In fact quantifiers play no role in SETNA as it is presented in this paper and can be omitted. The only reason SETNA was built as a FO theory was it intended extension into normative syllogistic in a similar way as Kalinowski has shown in his $\mathbf{K_2}$ system (cf. [Kalinowski, 1953]). This extension still remains to be carried out.

A0. *Axioms of PC and FOL with identity*

Coexistence is equivalence relation, i.e., it is reflexive (***A1***), symmetric (***A2***) and transitive (***A3***):

A1. $a \approx a$
A2. $a \approx b \to b \approx a$
A3. $a \approx b \wedge b \approx c \to a \approx c$

Intuitively, we say that two actions co-occur (for one and the same agent) if the agent has a physical possibility and an intellectual ability to carry out each of them individually (in the same situation), but not necessarily can perform both actions simultaneously.

" S " is a relation of parallel executability of two actions and is reflexive (***A4***) and symmetric (***A5***):

A4. aSa
A5. $aSb \to bSa$

A6 states that if the actions a and b are simultaneously executable, then they co-occur:

A6. $aSb \to a \approx b$

About the next two axioms can be said that each of them is a criterion of rational lawmaking. ***A7*** states that if some action is obligatory, then it is not prohibited:

A7. $Oa \to \neg Fa$

It does not allow that the same action in a system of norms, is at the same time obligatory and prohibited. ***A8*** says that if some two coexisting actions a and b are not simultaneously executable and a is obligatory, then b shall be prohibited:

A8. $\neg aSb \wedge a \approx b \to (Oa \to Fb)$

In other words, it does not allow to a norm-giver to order to the agent to carry out simultaneously two actions when they are for him/her physically or intellectually impossible to be done in parallel (see also ***T2*** and ***T8*** below).

In addition, we introduce the following definitions. ***Df 1*** defines the permitted action (" P ") as such, which is not prohibited:

Df 1. $Pa =_{df} \neg Fa$

Df 2 defines two actions as simultaneously non-executable (" N ") if and only if they can not be simultaneously carried out:

Df 2. $aNb =_{df} \neg aSb$

Action a is not regulated (" E ") if and only if it is neither obligatory nor prohibited:

Df 3. $Ea =_{df} \neg Oa \wedge \neg Fa$

We will show in the next section, that action which is not (explicitly) regulated is in fact permitted, so it is not entirely without normative qualification.

A few theses of SETNA:

T1. $Oa \rightarrow Pa$	*Any obligatory action is permitted.*
T2. $(aNb \wedge a \approx b \wedge Oa) \rightarrow Fb$	*If two coexisting actions are not executable in parallel and one of them is obligatory, then the other one is prohibited.*
T3. $Pa \vee Fa$	*Each action is either permitted or forbidden.*
T4. $Pa \equiv \neg Fa$	*Each action is permitted iff it is not prohibited.*
T5. $aNb \equiv \neg aSb$	*Two actions are simultaneously non-executable iff they cannot be carried out simultaneously.*
T6. $Ea \rightarrow Pa$	*Each action which is not regulated is permitted.*
T7. $(aNb \wedge bNc \wedge (Oa \vee Ob)) \rightarrow Fc$	*If a and b are non-executable and b and c are non-executable and (a is obligatory or b is obligatory), then c is prohibited.*
T8. $(a \approx b \wedge Oa \wedge Ob) \rightarrow aSb$	*If two action which coexist are obligatory then are simultaneously executable.*
T9. $Ea \equiv \neg Oa \wedge \neg Fa$	*Action is not regulated iff it is neither obligatory nor prohibited.*
T10. $(Oa \vee Ob) \rightarrow (Pa \vee Pb)$	*If a is obligatory or b is obligatory, then a is permitted or b is permitted.*

3. SETNA's model

SETNA's model is a structure:

$$\mathfrak{J} = <\varDelta, \cdot^{\varDelta}, \cdot^{\mathfrak{J}}>,$$

where $\varDelta$ is a structure which elements satisfy assumptions **Z1-Z10**, and $\cdot^{\varDelta}$, $\cdot^{\mathfrak{J}}$ are functions, which will be characterised as soon as $\varDelta$ is described.

3.1. Deontic structure *Δ*

$$\Delta = \langle D, Sim, Zak, Nak, Doz \rangle$$

is a structure satisfying the assumptions below.

$D = \{d, d_1, d_2, \ldots, d_n\}$ is a set of action tokens, which the agent may or is able to perform.[5] We assume that D is not empty:

Z1. $D \neq \varnothing$

Without committing to this condition, our model would allow for the existence of agent, who would not be able to perform any action, which obviously would lead to a conflict with known in philosophy (cf. [Bratman 1987, Davidson 1991, Searle 2001]) and in artificial intelligence (cf. [Franklin, Graesser 1996, Maes 1995, Russell, Norvig 1995, Wooldridge 2000]) definition of an agent as having ability to affect the environment (in which resides) by its actions.

A special place in the structure Δ occupies a set

Z2. $S = \{s, s_1, s_2, \ldots, s_k\} \subset 2^D$,

whose elements are certain subsets of a set of actions, possessing some special characteristics. Elements of a set S we would like to name *situations*, and about any action d belonging to a situation s we shall think that d is an action which the agent has a physical possibility and an intellectual ability to carry out in s. S is a subset of the power set of D which fulfils the conditions **Z3-Z10** described below.

Any action from the set D occurs only in one situation:

Z3. $\forall d[d \in D \rightarrow \exists! s(s \in S \wedge d \in s)]$

Thus for any $d \in D$ we always find such $s \in S$, such that d occurs in s, and such s is the only one. For instance having a set of actions $D = \{d_1, d_2\}$, **Z3** excludes the following set of situations: $S = \{\{d_1\}, \{d_1, d_2\}\}$. From the assumptions **Z1- Z3** it clearly follows that

W1. $S \neq \varnothing$

Proof. Z1 ensures that there is at least one element of a set D and **Z3** provides for each such element (exactly one) situation s, such that $d \in s$, what proves **W1**.

Moreover, for each situation there must be at least one action, which belongs to it:

Z4. $\forall s[s \in S \rightarrow \exists d(d \in D \wedge d \in s)]$

It is easy to see that **Z4** is equivalent to a statement that there is no empty situation (i.e. such a situation in which no action occurs):

W2. $\varnothing \notin S$

Proof. W2 derives directly from **Z4** stating non-emptiness of each element of a set S.

[5] For the simplicity reason, we consider, as we did so in section 2, that in the model there is only one agent. Therefore, all principles shall apply to one and the same agent.

What is worth attention is the fact that it is possible that $D \in S$, what has also interesting consequence:

W3. $D \in S \equiv \forall s(s \in S \to s = D)$

Proof. $(\to)$ Assume that $D \in S$. Because D includes all actions, therefore from **Z3** we get that for each action a set D is the only situation to which all actions belong. Given the above and lemma **W2**, D is the only element of a set S.

$(\leftarrow)$ Suppose that there is no such $s \in S$, that $s \neq D$. Then under **Z2**, **W1** and **W2**, we get that $D \in S$, what finally proves **W3**.

W3 concludes, therefore that $D \in S$ if and only if S is a singleton.

We say that two actions d_1 and d_2 co-occur if and only if they belong to the same situation:

D1. $Coex(d_1, d_2) =_{df} \exists s(s \in S \wedge d_1 \in s \wedge d_2 \in s)$

It is easy to see that

W4. *Coex is reflexive, symmetric and transitive.*

Proof. Reflexivity and symmetry of *Coex* are derived directly from the definition **D1**. Its transitivity we prove as follows. Suppose that for any d_1, d_2 and d_3, $Coex(d_1, d_2)$ and $Coex(d_2, d_3)$. From the definition of **D1** we have: $\exists s(s \in S \wedge d_1 \in s \wedge d_2 \in s)$ and $\exists s(s \in S \wedge d_2 \in s \wedge d_3 \in s)$. Eliminating existential quantifiers, we get that $s_1 \in S \wedge d_1 \in s_1 \wedge d_2 \in s_1$ and $s_2 \in S \wedge d_2 \in s_2 \wedge d_3 \in s_2$. Then from the fact that $d_2 \in s_1\, i\, d_2 \in s_2$ and from **Z3** it follows that $s_1 = s_2$. And therefore $s_1 \in S \wedge d_1 \in s_1 \wedge d_3 \in s_1$ and finally $Coex(d_1, d_3)$.

Some of the actions falling into the same situation, have the property that are executable simultaneously by the agent (e.g. when the agent is an experienced driver, such actions are *turn right* and *call by the phone*), while others (e.g., *turn right* and *turn left*) such property do not have. Let "$Sim(d_1,d_2)$" be the relation understood as "action d_1 is simultaneously executable with d_2". We say that two actions can be simultaneously executable only if both belong to the same situation:

Z5. $\forall d_1, d_2 \in D(Sim(d_1, d_2) \to Coex(d_1, d_2))$

It is worth noting that "*Sim*" is not equivalence relation, because it is not transitive. As a counterexample it is sufficient to take into account two pairs of actions simultaneously executable for some driver*: turn right – call by the phone* and *call by the phone – turn left* and notice that the pair of actions *turn right – turn left* is no longer simultaneously executable.

In addition:

Z6. *Sim is reflexive and symmetric.*

Then, let

D2. $Sim_d = \{d_1 \in D : Sim(d, d_1)\}$

be a set of these actions, which are simultaneously executable with *d*. Thus, a set

D3. $A_d = D \backslash Sim_d$

is a collection of these actions (belonging possibly to the same situation as *d* or to some other situations), which are not simultaneously executable with *d*. Then we define the relation saying that two actions are simultaneously non-executable:

D4. $Nonsim(d,d_1) \equiv d_1 \in A_d$

It is easy to prove that:

W5. $\forall d_1, d_2 \in D(Nonsim(d_1, d_2) \equiv \neg Sim(d_1, d_2))$

Proof. Lets take any two actions $d_1, d_2 \in D$. From **D4** we know that the formula $Nonsim(d_1, d_2)$ is equivalent to $d_2 \in A_{d_1}$, and then from **D3**, we get that it is also equivalent to $d_2 \in D \setminus Sim_{d_1}$. The last expression is equivalent to $d_2 \in D$ and $d_2 \notin Sim_{d_1}$, which in turn is equivalent under the **D2** to $\neg Sim(d_1, d_2)$. This finishes the proof of **W5**.

Then we introduce three subsets of *D*: *Zak, Nak* and *Doz*, which are the sets of actions being obligatory, prohibited and permitted, respectively. Membership to these sets shall be restricted by the principles described below.

If *d* is not prohibited, then it is also permitted:

Z7. $\forall d \in D(d \notin Zak \rightarrow d \in Doz)$

and if an action is prohibited, it is not permitted:

Z8. $\forall d \in D(d \in Zak \rightarrow d \notin Doz)$

From **Z7** and **Z8** it follows that

W6. $\forall d \in D(d \in Zak \equiv d \notin Doz)$

Proof. W6 is a direct consequence of **Z7** and **Z8**.

One can see, therefore that any action from the set *D* can be proved to a member either a set *Zak* or a set *Doz*:

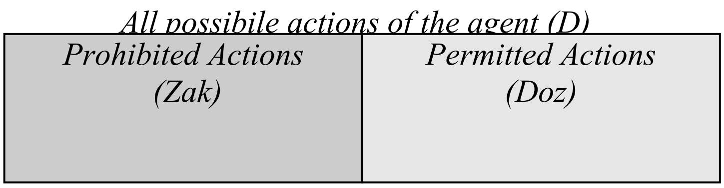

Figure 1

In addition, if *d* is obligatory, then it is not prohibited:

Z9. $\forall d \in D(d \in Nak \rightarrow d \notin Zak)$

As a consequence of **Z9** and **Z7** we get that a set of obligatory actions is a subset of a set of permitted ones:

W7. $\forall d \in D(d \in Nak \rightarrow d \in Doz)$

Proof. **Z9** guarantees that no obligatory action is prohibited, and **Z7** ensures that any action which is not prohibited is allowed. From those two assumptions **W7** directly follows.

If not all the actions from a set *Doz* are ordered, then **W7** may be illustrated as follows:

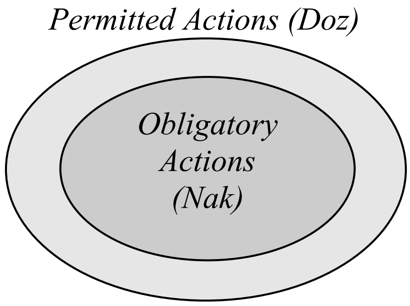

Figure 2

It is worth mentioning here that assumptions **Z7-Z9** and their consequences are the set-theoretical counterparts of widely accepted principles of deontic logic. In the form similar to presented in this paper they have been introduced for example by García Màynez (cf. [Kalinowski 1993, p.119-121]).

Finally, it appears also appropriate to introduce the following rule concerning actions which cannot be simultaneously carried out. The rule says that if some action is obligatory, then all actions simultaneously non-executable with it but belonging to the same situation, should be prohibited:

Z10. $\forall d_1, d_2 \in D\left(d_1 \in Nak \wedge Nonsim\left(d_1, d_2\right) \wedge Coex\left(d_1, d_2\right) \rightarrow d_2 \in Zak\right)$

This principle describes the relationship between norms of obligation and prohibition and the fact that some actions cannot be both physically or intellectually performed by the agent in the same situation.

Interesting consequence of this assumption says that if all the actions belonging to the same situation are obligatory, they all of them must be simultaneously executable:

W8. $\forall s \in S\left[\forall d \in D\left(d \in s \rightarrow d \in Nak\right) \rightarrow \forall d, d' \in D\left(d \in s \wedge d' \in s \rightarrow Sim\left(d, d'\right)\right)\right]$

Proof.

1. $s \in S$	assumption
2. $\forall d \in D\left(d \in s \rightarrow d \in Nak\right)$	assumption
3. $\neg\forall d, d' \in D\left(d \in s \wedge d' \in s \rightarrow Sim\left(d, d'\right)\right)$	*reductio*
4. $d_1 \in s \wedge d'_2 \in s \wedge \neg Sim\left(d_1, d'_2\right)$	FOL, $E\exists: 3$
5. $d_1 \in s \rightarrow d_1 \in Nak$	$E\forall: 2$
6. $d_1 \in Nak$	MP: 5,4
7. $Coex\left(d_1, d'_2\right)$	**D1**: 1,4
8. $Nonsim\left(d_1, d'_2\right)$	**W5**: 4
9. $d'_2 \in Zak$	MP: **Z10**,6,7,8

$10. d'_2 \in s \rightarrow d'_2 \in Nak$ $E\forall : 2$
$11. d'_2 \in Nak$ MP: 4,10
$12. d'_2 \notin Zak$ MP: **Z9**,11
contradiction: 9, 12

A special case of **W8** is the formula **W8'** saying, that if all the actions from a set *D* belong to one situation and are obligatory, then all of them must be also simultaneously executable:

W8'. $D = Nak \wedge D \in S \rightarrow \forall d,d' \in D(Sim(d,d'))$

Proof. **W8'** follows directly from **W8** and **W3**.

In other words, **W8'** states that it is enough to find in *D* two actions simultaneously non-executable to determine that either it is not the case that all actions in *D* are obligatory $(D \neq Nak)$ or D is not a situation $(D \notin S)$.

Two remaining elements of $\Im$ to be described are functions $\cdot^{\Delta}, \cdot^{\Im}$.

$\cdot^{\Delta}$ is an assignment function in $\Im$ characterized as below:

If $\boldsymbol{a}$ *is a constant,* $\boldsymbol{a}^{\Delta} \in D$
$O^{\Delta} = Nak$
$F^{\Delta} = Zak$
$S^{\Delta} = Sim$
$\approx^{\Delta} = Coex$

$\cdot^{\Im}$ is an interpretation function characterized as below:

if a *is a variable,* $a^{\Im} \in D$
if $\boldsymbol{a}$ *is a constant,* $\boldsymbol{a}^{\Im} = \boldsymbol{a}^{\Delta}$
if $R \in \{O, F, \approx, S\}$, $R^{\Im} = R^{\Delta}$

3.2. Soundness and completeness of SETNA

Below we provide satisfaction conditions for formulae of SETNA. The satisfaction conditions for the quantifiers and the operators of PC are standard.

$\Im \models t_1 \approx t_2$ *iff* $Coex(t_1^{\Im}, t_2^{\Im})$,
$\Im \models t_1 S t_2$ *iff* $Sim(t_1^{\Im}, t_2^{\Im})$,
$\Im \models Ft$ *iff* $t^{\Im} \in Zak$,
$\Im \models Ot$ *iff* $t^{\Im} \in Nak$,

As the consequences of those conditions we get:

$\Im \models t_1 N t_2$ *iff* $Nonsim(t_1^{\Im}, t_2^{\Im})$,
$\Im \models Pt$ *iff* $t^{\Im} \in Doz$,

Theorem 1 (Soundness). *All SETNA theses are true in the model* $\Im$*:*

$$\text{if} \vdash \phi, \text{ then } \models \phi$$

Proof. It is to be shown that all the axioms of the theory are valid and that truth is inherited by the rules of the theory.

- The truth of the axioms **A1-A3** derive directly from the satisfaction conditions and **W4**.

- The truth of axioms **A4** and **A5** derives directly from the conditions of satisfaction and assumption **Z6**, i.e., from reflexivity and symmetry of *Sim*.
- $\mathfrak{I} \models a\, S\, b \to a \approx b$
  Lets assume that $\mathfrak{I} \models a\ S\ b$. This assumption is equivalent to *Sim*($a^{\mathfrak{I}}$, $b^{\mathfrak{I}}$). From **Z5** and our assumption we get that *Coex*($a^{\mathfrak{I}}$, $b^{\mathfrak{I}}$), from which it follows that $\mathfrak{I} \models a \approx b$.
- $\mathfrak{I} \models Oa \to \neg Fa$
  Lets assume that $\mathfrak{I} \models Oa$ and $\mathfrak{I} \models Fa$. From this assumption we obtain that $a^{\mathfrak{I}} \in Nak$ and $a^{\mathfrak{I}} \in Zak$. Then from **Z9** we get that $a^{\mathfrak{I}} \notin Zak$, what leads to contradiction.
- $\mathfrak{I} \models \neg aSb \wedge a \approx b \to (Oa \to Fb)$
  Lets assume that $\mathfrak{I} \models \neg\, a\ S\ b$, $\mathfrak{I} \models a \approx b$, $\mathfrak{I} \models Oa$ and not $\mathfrak{I} \models Fb$. From those assumptions we get that *Nonsim*($a^{\mathfrak{I}}$, $b^{\mathfrak{I}}$), *Coex*($a^{\mathfrak{I}}$,$b^{\mathfrak{I}}$), $a^{\mathfrak{I}} \in Nak$ and $b^{\mathfrak{I}} \notin Zak$. Under **Z10** from the fact that *Nonsim*($a^{\mathfrak{I}}$, $b^{\mathfrak{I}}$), *Coex*($a^{\mathfrak{I}}$,$b^{\mathfrak{I}}$) and $a^{\mathfrak{I}} \in Nak$, it follows that $b^{\mathfrak{I}} \in Zak$, what leads to contradiction.

The proof finishes the remark that the rule of *Modus Ponens* leads from true formulae to true formulae.

Theorem 2 (Completeness). *All the true formulae of SETNA are its theses, i.e.,*

(*) *if* $\models \phi$, *then* $\vdash \phi$

Proof. Lets start from two terminological points. By "$\Phi \models \phi$" we shall understand that each model for the set of formulae Φ is also a model for ϕ but by "$\Phi \vdash \phi$" that ϕ is derivable from the set of formulae Φ (and the theses of SETNA). Then in order to prove (*) it is enough to prove that

(**) *for any set of formulae* Φ *and any formula* ϕ, *if* $\Phi \models \phi$, *then* $\Phi \vdash \phi$

and notice that (*) follows from (**) for $\Phi = \varnothing$.

Then in order to prove (**), lets assume that $\Phi \models \phi$ and not $\Phi \vdash \phi$. From $\Phi \models \phi$ it follows that (i) there is no model for the set of formulae $\Phi \cup \{\neg\phi\}$ [cf. 4.4 Lemma, Ebbinghaus et al., 1994], but from the second assumption we get that (ii) $\Phi \cup \{\neg\phi\}$ is consistent (cf. Lemma 7.6 (a), [Ebbinghaus et al. 1994]). Consequences (i) and (ii) are in contradiction with the fact that

(***) *For each consistent set of formulae there is a model.*

Fact (***) we shall prove below. As shown in [Ebbinghaus et al. 1994] there is a "natural" way to do this. The sketch of the method we shall show below.

Let Φ be a set of formulae of SETNA such that

a. Φ is consistent
b. for all formulae ϕ either $\phi \in \Phi$ or $\neg\phi \in \Phi$
c. for all formulae $\exists x\phi$, there is a term t such that $(\exists x\phi \to \phi[t/x]) \in \Phi$

Lemma 1 (Consistency). SETNA *is consistent.*

Proof.
It is easy to prove the lemma by the method of interpretation. The variables of the system shall be interpreted as variables of PC and predicates S, $\approx$, O and F, respectively as equivalence, equivalence, assertion and negation. Under this interpretation all the axioms of SETNA are theses of PC with quantifiers.

Lemma 2.
Any consistent set of formulas can be extended to a set which satisfies the conditions b and c.

Proof.
See the proofs of lemmas 2.1, 2.2 and 3.1 and 3.2 in [Ebbinghaus et al. 1994, p.81-86].

Lemma 3.
Canonical structure $\Delta^{\Phi} = <D^{\Phi}, Nak^{\Phi}, Zak^{\Phi}, Doz^{\Phi}, Sim^{\Phi}>$ - constructed from the formulae of the language of SETNA as below – is Δ –structure.

Let $t_1 \sim t_2$ *iff* $t_1 = t_2 \in \Phi$. It is easy to see that $\sim$ is equivalence relation. By $|t|_{\sim}$ we shall mean the equivalence class of set of terms by $\sim$ to which t belongs. Then $\Im = \langle \Delta^{\Phi}, \cdot^{\Delta^{\Phi}}, \cdot^{\Im^{\Phi}} \rangle$, is a model such that $\cdot^{\Im^{\Phi}}$ is characterized in the same way as $\cdot^{\Im}$:

$$a^{\Im^{\Phi}} = |a|_{\sim}$$

$$a^{\Im^{\Phi}} = |a|_{\sim}$$

$$O^{\Im^{\Phi}} = Nak^{\Phi}$$

$$F^{\Im^{\Phi}} = Zak^{\Phi}$$

$$P^{\Im^{\Phi}} = Doz^{\Phi}$$

$$S^{\Im^{\Phi}} = Sim^{\Phi}$$

$$\approx^{\Im^{\Phi}} = Coex^{\Phi}$$

$D^{\Phi} = \{|t|_{\sim} : t \textit{ is a term}\}$ is a quotient set of the set of terms by $\sim$. Sometimes we shall refer to the elements of this set by d_1^{Φ}, d_2^{Φ}, ... Then structure Δ^{Φ}, is built on the basis of the domain D^{Φ} as follows. First we define: $|t_1|_{\sim} \cong |t_2|_{\sim}$ *iff* $t_1 \approx t_2 \in \Phi$. By $||t|_{\sim}|_{\cong}$ we shall mean the equivalence class of D^{Φ} by $\cong$ to which $|t|_{\sim}$ belongs. $S^{\Phi} = \{||t|_{\sim}|_{\cong} : |t|_{\sim} \in D^{\Phi}\}$ is a quotient set of D^{Φ} by $\cong$. We shall refer to the elements of S^{Φ} by s_1^{Φ}, s_2^{Φ}, ... Then we introduce the remaining sets from Δ^{Φ}:

$$Nak^{\Phi} = \{|t|_{\sim} : Ot \in \Phi\}$$

$$Zak^{\Phi} = \{|t|_{\sim} : Ft \in \Phi\}$$

$$Doz^{\Phi} = D^{\Phi} \setminus Zak^{\Phi}$$

$Sim^{\Phi}(|t_1|_{\sim}, |t_2|_{\sim})$ *iff* $t_1\, S\;\; t_2 \in \Phi$

$Coex^{\Phi}(d_1^{\Phi}, d_2^{\Phi}) =_{df} \exists s^{\Phi}\left(s^{\Phi} \in S^{\Phi} \wedge d_1^{\Phi} \in s^{\Phi} \wedge d_2^{\Phi} \in s^{\Phi}\right)$

Proof.
The proof of lemma 3 is easy but time consuming. In order to prove it, it is enough to show that structure Δ^{Φ} satisfies properties **Z1** – **Z10**. We shall omit this proof.

Lemma 4.
(a) *for any term t,* $t^{\Im^{\Phi}} = |t|_{\sim}$
(b) *for any atomic formula* ϕ,

$$\Im^{\Phi} \models \phi \text{ iff } \phi \in \Phi$$

(Satisfaction conditions are defined in the similar way as before).

Proof.
Part (a) of the proof is obvious. Part (b) we prove by induction in the following way:

For $\phi = (t_1 = t_2)$:
$\Im^{\Phi} \models t_1 = t_2$ *iff* $t_1^{\Im^{\Phi}} = t_2^{\Im^{\Phi}}$ *iff* $|t_1|_{\sim} = |t_2|_{\sim}$ *iff* $t_1 \sim t_2$ *iff* $t_1 = t_2 \in \Phi$.

For $\phi = Ot$ *(in the similar way we prove for* $\phi = Ft$*):*
$\Im^{\Phi} \models Ot$ *iff* $t^{\Im^{\Phi}} \in Nak^{\Phi}$ *iff* $|t|_{\sim} \in Nak^{\Phi}$ *iff* $Ot \in \Phi$.

For $\phi = t_1 \approx t_2$:
$\Im^{\Phi} \models t_1 \approx t_2$ *iff* $Coex^{\Phi}\left(t_1^{\Im^{\Phi}}, t_2^{\Im^{\Phi}}\right)$ *iff* $\exists s^{\Phi}\left(s^{\Phi} \in S^{\Phi} \wedge t_1^{\Im^{\Phi}} \in s^{\Phi} \wedge t_2^{\Im^{\Phi}} \in s^{\Phi}\right)$ *iff*
$\exists s^{\Phi}\ (s^{\Phi} \in S^{\Phi} \wedge |t_1|_{\sim} \in s^{\Phi} \wedge |t_2|_{\sim} \in s^{\Phi})$ *iff*
$\exists s^{\Phi}\ (s^{\Phi} \in S^{\Phi} \wedge s^{\Phi} = ||t_3|_{\sim}|_{\cong} \wedge t_1 \approx t_3 \in \Phi \wedge t_2 \approx t_3 \in \Phi)$ *iff* $t_1 \approx t_2 \in \Phi$.

For $\phi = t_1\, S\, t_2$:
$\Im^{\Phi} \models t_1\, S\, t_2$ *iff* $Sim^{\Phi}\left(t_1^{\Im^{\Phi}}, t_2^{\Im^{\Phi}}\right)$ *iff* $Sim^{\Phi}(|t_1|_{\sim}, |t_2|_{\sim})$ *iff* $t_1\, S\;\; t_2 \in \Phi$.

Henkin's Theorem.
Let Φ *be a set of formulae satisfying conditions a-c in lemma 1. Then for any* ϕ*:*

$$(\#)\quad \Im^{\Phi} \models \phi \text{ iff } \phi \in \Phi$$

Proof.
We prove this theorem by induction, depending on the complexity of the formula ϕ. If ϕ is an atomic formula, then (#) holds under lemma 4. For $\phi = \neg\psi$, $\phi = \psi \vee \chi$ and $\phi = \exists x\psi$, the prove is standard (cf. Henkin's Theorem in [Ebbinghaus et al. 1994]).

We have shown that for any consistent set of formulae Φ there is a model, which is based on Δ-structure, what entails by the lemmas and theorems proved above that SETNA is complete with respect to model $\Im$.

4. SETNA in action

In this section an application of SETNA will be presented. As it has been mentioned in the introduction, the SETNA's theses can be used as a basis for the correct reasoning for instance in driving situations. Below we shall consider three questions, which certainly are familiar to those who were trying to get driving license in Poland in the recent years, because they are derived from driving theory test questions. Each of the questions consists of a photo illustrating a driving situation and a problem to be solved consisting of three of four norms which are to be recognized as being in force or not in the situation.

Situation 1 (s_1)

Figure 3

In situation $\mathbf{s_1}$ a driver:
A – should turn right (at the crossroads)
B – is permitted to turn right (at the crossroads)
C – is not permitted to turn right (at the crossroads)
D – is permitted to go straight

In $\mathbf{s_1}$ a norm expressed by a sign B-21: *forbiddance of turning left* is in force.

Let *sp, sl, jp,* be the names of actions such that $sp^{\Im}$ = *turn right*, $sl^{\Im}$ = *turn left*, $jp^{\Im}$ = *go straight*.

1. We establish which of those three actions can be performed simultaneously, and which cannot. In fact any two of them cannot be carried out in parallel, what is expressed in the language of SETNA as follows:

$$sp\ N\ sl,\ sp\ N\ jp,\ sl\ N\ jp$$

2. We find the norms expressed by the traffic signs:

Fsl (this norm is expressed by the sign B-21)

and those actions which are not regulated:

Esp, Ejp

3. We check:

– whether the sentences of the test are inferred from the assumptions made, the axioms and the rules of SETNA – in this case we accept them as the correct answers

– whether they attached to the assumptions lead to contradiction – in this case they are incorrect answers.

We have therefore **A-D** options which expressed in the language of SETNA take the following forms:

A – *Osp*

B – *Psp*

C – $\neg$*Psl*

D – *Pjp*

The answer **A** leads to contradiction with assumption *Esp* and thesis ***T9***, **B** follows from assumption *Esp* and ***T6***, **C** follows from assumption *Fsl* and ***T4***, and **D** can be inferred from *Esp* and ***T6***. Thus the correct answers are: ***B***, ***C*** and ***D***.

Situation 2 (s_2)

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In situation $\mathbf{s_2}$ a driver:
A – is not permitted to turn right
B – should turn right
C – is permitted to turn left

In situation $\mathbf{s_2}$ a norm expressed by the sign C-2: *Obligation to turn right (after the sign)* is in force.

Let *sp, sl* be the names of actions such that $sp^{\Im}$ = *turn right*, $sl^{\Im}$ = *turn left*. We shall proceed now in accordance with the "algorithm" set out above:

1. We establish which actions can be performed simultaneously, and which cannot:

$$sp\ N\ sl$$

2. We find the norms expressed by the traffic signs:

$$Osp$$

and those actions which are not regulated:

there aren't any

It is worth noting that it cannot be said that action *sl* is not regulated, because from the obligation to carry out *sl* and the fact that it cannot be done simultaneously with *sl,* under ***T2*** it follows that *sl* is prohibited**.** Thus by ***T2*** and the facts that *sp N sl* and *Osp* it follows that

$$Fsl$$

3. We shall express the sentences **A-C** into SETNA's language, in the similar way as we have done in the earlier example:

A – *¬Psp (leads to contradiction with the assumption Osp and* ***T1****)*

B – *Osp (follows from Osp)*

C – *Psl (leads to contradiction with the assumptions Osp, sp N sl and theses* ***T2****,* ***T4****)*

Thus the correct answers is only ***B***.

Situation 3 (s_3)

Figure 5

In situation $\mathbf{s_3}$ a driver:
A – is not permitted to turn right or left
B – should turn back
C – should turn right or left

In situation $\mathbf{s_3}$ a norm expressed by the sign C-8: *Obligation to turn right or left* is in force.

Let *sp, sl, z* be the names of actions such that $sp^{\Im}$ = *turn right*, $sl^{\Im}$ = *turn left*, $z^{\Im}$ = *turn back*

1. We establish which operations can be performed simultaneously, and which cannot:

$$sp\ N\ sl,\ sp\ N\ z,\ sl\ N\ z$$

2. We find the norms expressed by the traffic signs and those actions which are not regulated:

$$Osp \vee Osl$$

It cannot be said that action z is not regulated, because from the facts and norms: $N\,z$, $sl\,N\,z$, $Osp \vee Osl$ *by* ***T7*** it follows that

$$Fz$$

3. We shall express the sentences **A-C** into SETNA's language, in the similar way as we have done in the earlier examples:

A – $\neg(Psp \vee Psl)$ *(leads to contradiction with the assumption* $Osp \vee Osl$ *and* ***T10****)*

B – *Oz (leads to contradiction with the assumptions sp N z, sl N z,* $Osp \vee Osl$ *, thesis* ***T7*** *and axiom* ***A7****)*

C – $Osp \vee Osl$ *(follows from* $Osp \vee Osl$ *)*

Thus the correct answer is only ***C****.*

5. Conclusion

In this work a theory of norms and actions was presented, which due to its initial phase of development was named "simple". Its particular feature is that it contains the laws describing the relations between the non-deontic properties of actions and their deontic properties. Special attention deserves the principle establishing the relationship between simultaneous (non-)execution of two or more actions and the norms of obligation or prohibition (cf. **A8**, ***T2*** and ***T8***).

It was also shown the applicability of SETNA in solving some questions concerning driving situations. Obviously this application does not exhaust a list of all the possible uses of this theory. SETNA can be successfully used as an aid in the creation of any systems of norms and as a tool for verifying the systems of norms already existing. A study of this issue, however, deserves a separate work.

Since 2008 when SETNA was published for the first time a few important extensions of it have been made. First of all the language of SETNA was changed from FOL to modal setting. All the properties of SETNA has been expressed into *deontic action logic based on Boolean algebra*. This framework allowed for introducing internal operators such as negation of action, indeterministic choice or parallel execution, essentially increasing expressing power of the basic theory. In [Trypuz, Kulicki 2009] a metalogical systematization in the area of deontic action logic based on Boolean algebra was provided, placing SETNA and its extensions among other systems. In particular it has been shown that the differences among the systems in question lie in two aspects: the level of closedness of a deontic action logic and the possibility of performing no action at all. In [Trypuz, Kulicki 2010] it was shown that the existing definitions of obligation in the systems of deontic action logic based on Boolean algebra are not acceptable due to their unintuitive interpretation or paradoxical consequences. As a solution it was proposed an axiomatic characterization of obligation with an adequate class of models. Finally in [Trypuz, Kulicki 2011] a formal system motivated by SETNA specific methodology of creating norms was presented. According to the methodology, a norm-giver before establishing a set of norms should create a picture of the agent by creating his repertoire of actions. Then, knowing what the agent can do in particular situations, the norm-giver regulates these actions by assigning deontic qualifications to each of them. In this extension the concept of situation has been essentially modified. In the "original" SETNA presented in this paper a situation is a set of actions, whereas in [Trypuz, Kulicki 2011] it is understood much like "possible world" in standard modal logic.

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